Superintegrable quantum oscillator and Kepler-Coulomb systems on curved spaces

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Abstract

An overview of maximally superintegrable classical Hamitonians on spherically symmetric spaces is presented. It turns out that each of these systems can be considered either as an oscillator or as a Kepler–Coulomb Hamiltonian. We show that two possible quantization prescriptions for all these curved systems arise if we impose that superintegrability is preserved after quantization, and we prove that both possibilities are gauge equivalent.

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1 Bertrand spacetimes

Bertrand's theorem asserts that any three-dimensional (3D) spherically symmetric natural Hamiltonian system

$$H = \frac{1}{2}\mathbf{p}^2 + V(|\mathbf{q}|)$$

that has a stable circular trajectory passing through each point of $\mathbb{R}^3$ and all whose bounded trajectories are closed is either a harmonic oscillator, $V(r) = \omega^2\mathbf{q}^2 + a$, or a Kepler–Coulomb (KC) system, $V(r) = \frac{k}{|\mathbf{q}|} + b$. An extension of such a classical theorem was found by Perlick [1] in a General Relativity framework. Consider a $(3+1)$D spherically symmetric spacetime $(\mathcal{M} \times \mathbb{R}, \eta)$, where $\mathcal{M}$ is a 3-manifold. Then the Lorentzian metric η can be written as

$$\eta = h(r')^2\,\mathrm{d}r'^2 + r'^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\big) - \frac{\mathrm{d}t^2}{V(r')} = g - \frac{\mathrm{d}t^2}{V(r')},$$

where g is a Riemannian metric on $\mathcal{M}$ and $h(r'), V(r')$ are smooth functions. A *trajectory* in the spacetime is the projection of an inextendible timelike geodesic to a constant time leaf $\mathcal{M}\{t_0\}$. The Lorentzian (3+1)-manifold $(\mathcal{M} \times \mathbb{R}, \eta)$ is a *Bertrand spacetime* if it is verified that: (i) there is a circular ($r' =$ constant) trajectory passing through each point of $\mathcal{M}$, and (ii) such circular trajectories are stable. Under these assumptions, Perlick undertook the classification of all Bertrand spacetimes finding two multiparametric families of metrics by obtaining explicitly the functions $h(r'), V(r')$.

These results can be extended to arbitrary dimension and expressed in a conformally flat form. Let us consider an $(N+1)$D spherically symmetric spacetime $(\mathcal{M} \times \mathbb{R}, \eta)$, where hereafter $\mathcal{M}$ is an N-manifold. The Lorentzian metric η can be written as

$$\eta = g - \frac{\mathrm{d}t^2}{V(r)}, \qquad g = f(|\mathbf{q}|)^2\,\mathrm{d}\mathbf{q}^2 = f(r)^2(\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2), \tag{1.1}$$

where $\mathbf{q} = (q_1, \dots, q_N)$, $\mathrm{d}\mathbf{q}^2 = \sum_{i=1}^N \mathrm{d}q_i^2$, $r = |\mathbf{q}| = \sqrt{\mathbf{q}^2}$ and $\mathrm{d}\Omega^2$ is the standard metric on the unit $(N-1)$D sphere. Hence, g defines a Riemannian metric on $\mathcal{M}$ with $f(r)$ playing the role of *conformal factor* of the Euclidean metric $g_0 = \mathrm{d}\mathbf{q}^2$. The scalar curvature of g is generally nonconstant and turns out to be

$$R = -(N-1)\left(\frac{(N-4)f'(r)^2 + f(r)\left(2f''(r) + 2(N-1)r^{-1}f'(r)\right)}{f(r)^4}\right). \tag{1.2}$$

By starting from Perlick's classification of $(3+1)$D Bertrand spacetimes, if we change the initial radial coordinate in the form $r' = \mathcal{F}(r)$ and we accordingly transform the initial Perlick's parameters, then it can be proven through very cumbersome computations that the metric η of an $(N+1)$D Bertrand spacetime belongs to one of the following classes [2]:

• **Type I**. This is a *three-parametric* family of metrics depending on $(\beta; \kappa, \xi)$ where β is a positive rational number and κ, ξ are real constants:

$$\eta_{\mathrm{I}} = \frac{1}{r^2\left(r^{-\beta} + \kappa r^{\beta}\right)^2}\,\mathrm{d}\mathbf{q}^2 - \frac{\mathrm{d}t^2}{(r^{-\beta} - \kappa r^{\beta}) + \xi}. \tag{1.3}$$

• **Type II**. This is a *four-parametric* family depending on $(\gamma; \lambda, \delta, \chi)$ where γ is a positive rational number and λ^2, δ, χ are real constants:

$$\eta_{\mathrm{II}} = \frac{\left(r^{-2\gamma} + \lambda^2 r^{2\gamma} - 2\delta\right)}{r^2\left(r^{-2\gamma} - \lambda^2 r^{2\gamma}\right)^2}\,\mathrm{d}\mathbf{q}^2 - \frac{\mathrm{d}t^2}{\left(r^{-2\gamma} + \lambda^2 r^{2\gamma} - 2\delta\right)^{-1} + \chi}. \tag{1.4}$$

The sets of parameters $(\beta;\kappa,\xi)$ and $(\gamma;\lambda,\delta,\chi)$ can be, in fact, regarded as *deformation parameters* of the *flat* Minkowskian metric, since this is recovered by a contraction limit of (1.3) and (1.4) as follows:

$$
\begin{array}{llllllllll}
\eta_{\rm I} & = \mathrm{d}\mathbf{q}^2 - \mathrm{d}t^2 & \text{for} \ \beta & = 1, & \kappa & = 0, & \xi & = 1/\varepsilon^2; & t \to t/\varepsilon, & \varepsilon \to 0. \\
\eta_{\rm II} & = \mathrm{d}\mathbf{q}^2 - \mathrm{d}t^2 & \text{for} \ \gamma & = 1, & \lambda & = \delta = 0, & \chi & = 1/\varepsilon^2; & t \to t/\varepsilon, & \varepsilon \to 0.
\end{array}
$$

2 Classical Bertrand Hamiltonians

Timelike geodesics in an $(N+1)$D spacetime with metric η (1.1) are naturally related to trajectories of the ND classical Hamiltonian in $\mathcal{M}$ given by

$$
H = \frac{\mathbf{p}^2}{2f(|\mathbf{q}|)^2} + V(|\mathbf{q}|) = \frac{p_r^2 + r^{-2}\mathbf{L}^2}{2f(r)^2} + V(r), \tag{2.5}
$$

where $\mathbf{p} = (p_1, \ldots, p_N)$ and p_r are, in this order, the conjugate momenta of the coordinates $\mathbf{q}$ and r, while $\mathbf{L}$ is the total angular momentum.

Thus, Perlick's study of Bertrand spacetimes provides a complete classification of the central potentials $V(r)$ and spherically symmetric metrics for which the statement of the classical Bertrand theorem remains true. Moreover, we showed that these Hamiltonian systems are still superintegrable [3]. Therefore, by considering the classification of Bertrand spacetimes (1.3) and (1.4), we write the corresponding *classical Bertrand Hamiltonians* which are cast into the following two families [2]:

• **Type I**. *Two-parametric* family of Hamiltonians depending on $(\beta;\kappa)$:

$$
H_{\rm I} = \frac{1}{2} r^2 \left(r^{-\beta} + \kappa r^{\beta}\right)^2 \mathbf{p}^2 + A\left(r^{-\beta} - \kappa r^{\beta}\right). \tag{2.6}
$$

• **Type II**. *Three-parametric* family depending on $(\gamma;\lambda,\delta)$:

$$
H_{\rm II} = \frac{r^2\left(r^{-2\gamma} - \lambda^2 r^{2\gamma}\right)^2}{2\left(r^{-2\gamma} + \lambda^2 r^{2\gamma} - 2\delta\right)}\, \mathbf{p}^2 + \frac{B}{\left(r^{-2\gamma} + \lambda^2 r^{2\gamma} - 2\delta\right)}. \tag{2.7}
$$

We remark that the parameters ξ, χ, which are rather relevant in the metrics, become additive constants in the Hamiltonians. The coupling constants A, B of the potentials do not appear in the metrics since they can be set equal to 1 through a time scaling. Note also that Bertrand Hamiltonians are, as expected, generalizations (deformations) of the flat harmonic oscillator and KC systems, which are recovered as particular cases:

$$
\begin{array}{llllll}
H_{\rm I} & = \frac{1}{2}\mathbf{p}^2 + A/r & \text{for} \ \beta & = 1, & \kappa & = 0. \\
H_{\rm II} & = \frac{1}{2}\mathbf{p}^2 + Br^2 & \text{for} \ \gamma & = 1, & \lambda & = \delta = 0.
\end{array}
$$

The main properties of Bertrand Hamiltonians are the following:

(i) *Superintegrability.* The spherical symmetry of any classical Hamiltonian of the type (2.5) provides $2N-3$ integrals of motion which are quadratic in the momenta. *Maximally superintegrable* systems are distinguished cases for which an additional constant of the motion exists (which is a component of a Runge–Lenz N-vector), so that H is endowed with $2N-2$ functionally independent integrals. This has been explicitly proved for 3D Bertrand Hamiltonians [3], finding that the integrals of motion are not, in general, quadratic in the momenta (such higher order is determined by β, γ).

(ii) *Geometric interpretation.* The *intrinsic KC and oscillator potentials* on the ND space $\mathcal{M}$ are defined by means of the radial symmetric Green function $U(r)$ of the Riemannian metric g (1.1) as

$$U(r) = \int^r \frac{\mathrm{d}r}{r^2 f(r)}, \qquad \mathcal{U}_{\mathrm{KC}}(r) := A\,U(r), \qquad \mathcal{U}_{\mathrm{O}}(r) := \frac{B}{U(r)^2}.$$

If we apply these definitions to the two families of Bertrand Hamiltonians it can be shown [4] that type I (2.6) and type II Hamiltonians (2.7) always define, in this order, intrinsic KC systems and oscillator potentials on curved spaces. Consequently, the extension of the classical Bertrand's theorem to spherically symmetric spaces only provides either oscillators or KC systems.

(iii) *Stäckel transform.* Both types of Bertrand Hamiltonians are equivalent via the Stäckel transform [5]. If we denote (2.6) as $H_{\mathrm{I}} = T_{\mathrm{I}} + V_{\mathrm{I}}$ and consider $H_{0,\mathrm{I}} = T_{\mathrm{I}} + B$ and $U_{\mathrm{I}} = V_{\mathrm{I}}/A + C$, then $H_{0,\mathrm{I}}/U_{\mathrm{I}} \equiv H_{\mathrm{II}}$ provided that

$$\beta = 2\gamma, \quad \kappa = -\lambda^2, \quad C = -2\delta.$$

Conversely, Hamiltonians of type I can be obtained from those of type II, so establishing a *"duality"* between curved oscillators and KC systems [2].

We illustrate the above results by displaying in Table 1 some relevant specific Bertrand Hamiltonians that arise for $\beta = 1, 2 \leftrightarrow \gamma = \frac{1}{2}, 1$; all of them have quadratic integrals [2]. The remaining parameters, κ and (λ, δ) are related with the curvature of the space. In each row we present a KC system with its Stäckel equivalent ("dual") oscillator. The names "Euclidean", "Taub–NUT"..., refer to the underlying Riemannian space. Notice that the KC system on the *three* spaces of *constant* sectional curvature κ ($>, <, = 0$ for the sphere, hyperbolic space, Euclidean space, respectively) is recovered in the cases 1A and 1B of type I ($\beta = 1$), meanwhile their corresponding oscillator potential appears in the cases 2A.1 and 2B.1 of type II ($\gamma = 1$), with $\lambda = \delta$ now playing the role of the constant curvature.

3 Quantum Bertrand Hamiltonians

In order to obtain the quantization of the classical Bertrand Hamiltonians (2.6) and (2.7), the ordering ambiguity coming from the kinetic term can be fixed by imposing the resulting quantum Hamiltonians $\hat{\mathcal{H}}$ to be maximally superintegrable. This means that the classical integrals are promoted to $(2N-2)$ algebraically independent operators that commute with $\hat{\mathcal{H}}$. We consider the quantum position and momenta operators, $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}$, such that $[\hat{q}_i, \hat{p}_j] = \mathrm{i}\hbar\delta_{ij}$, $\hat{q}_i = q_i$, $\hat{p}_i = -\mathrm{i}\hbar\frac{\partial}{\partial q_i}$, and we present two quantization prescriptions that fulfil the superintegrability condition [6]: the so-called Schrödinger quantization and the conformal Laplace–Beltrami (LB) one.

• *"Direct" or Schrödinger quantization* [7]. This prescription provides maximally superintegrable Hamiltonians whose spectra presents, as expected, accidental degeneracy [8]. They are given by

$$\hat{\mathcal{H}}_{\mathrm{I}} = \frac{1}{2}\hat{\mathbf{q}}^2 \left(|\hat{\mathbf{q}}|^{-\beta} + \kappa|\hat{\mathbf{q}}|^{\beta}\right)^2 \hat{\mathbf{p}}^2 + A\Big(|\hat{\mathbf{q}}|^{-\beta} - \kappa|\hat{\mathbf{q}}|^{\beta}\Big),$$

$$\hat{\mathcal{H}}_{\mathrm{II}} = \frac{\hat{\mathbf{q}}^2\left(|\hat{\mathbf{q}}|^{-2\gamma} - \lambda^2|\hat{\mathbf{q}}|^{2\gamma}\right)^2}{2\left(|\hat{\mathbf{q}}|^{-2\gamma} + \lambda^2|\hat{\mathbf{q}}|^{2\gamma} - 2\delta\right)}\,\hat{\mathbf{p}}^2 + \frac{B}{\left(|\hat{\mathbf{q}}|^{-2\gamma} + \lambda^2|\hat{\mathbf{q}}|^{2\gamma} - 2\delta\right)}.$$

Table 1: Examples of N-dimensional classical Bertrand Hamiltonians.

Type I: KC Hamiltonians $(\beta;\kappa)$	Type II: Oscillator Hamiltonians $(\gamma;\lambda,\delta)$
• 1A $(\beta=1;\kappa=0)$ Euclidean $H=\dfrac{1}{2}\mathbf{p}^2+\dfrac{A}{r}$	• 1A $(\gamma=\frac{1}{2};\lambda=0,\delta)$ Taub–NUT $H=\dfrac{r}{2(1-2\delta r)}\,\mathbf{p}^2+\dfrac{Br}{(1-2\delta r)}$
• 1B $(\beta=1;\kappa)$ Sphere/Hyperbolic $H=\dfrac{1}{2}(1+\kappa r^2)^2\mathbf{p}^2+A\,\dfrac{(1-\kappa r^2)}{r}$	• 1B $(\gamma=\frac{1}{2};\lambda,\delta)$ Darboux IV $H=\dfrac{(1-\lambda^2r^2)^2r}{2(1+\lambda^2r^2-2\delta r)}\,\mathbf{p}^2+\dfrac{Br}{(1+\lambda^2r^2-2\delta r)}$
• 2A $(\beta=2;\kappa=0)$ $H=\dfrac{\mathbf{p}^2}{2r^2}+\dfrac{A}{r^2}$	• 2A.1 $(\gamma=1;\lambda=0,\delta=0)$ Euclidean $H=\dfrac{1}{2}\mathbf{p}^2+Br^2$ • 2A.2 $(\gamma=1;\lambda=0,\delta)$ Darboux III $H=\dfrac{\mathbf{p}^2}{2(1-2\delta r^2)}+\dfrac{Br^2}{(1-2\delta r^2)}$
• 2B $(\beta=2;\kappa)$ $H=\dfrac{(1+\kappa r^4)^2}{2r^2}\,\mathbf{p}^2+A\,\dfrac{(1-\kappa r^4)}{r^2}$	• 2B.1 $(\gamma=1;\lambda,\delta=\lambda)$ Sphere/Hyperbolic $H=\dfrac{1}{2}(1+\lambda r^2)^2\mathbf{p}^2+\dfrac{Br^2}{(1-\lambda r^2)^2}$ • 2B.2 $(\gamma=1;\lambda,\delta)$ $H=\dfrac{(1-\lambda^2r^4)^2}{2(1+\lambda^2r^4-2\delta r^2)}\,\mathbf{p}^2+\dfrac{Br^2}{(1+\lambda^2r^4-2\delta r^2)}$

• *Conformal LB quantization.* The quantum free Hamiltonian could be constructed by means of the usual LB operator Δ_{LB} for the metric g (1.1):

$$\hat{\mathcal{H}}_{\mathrm{LB,I}}=-\tfrac{1}{2}\hbar^2\Delta_{\mathrm{LB}}+A\Big(|\hat{\mathbf{q}}|^{-\beta}-\kappa|\hat{\mathbf{q}}|^{\beta}\Big),$$
$$\hat{\mathcal{H}}_{\mathrm{LB,II}}=-\tfrac{1}{2}\hbar^2\Delta_{\mathrm{LB}}+B\big(|\hat{\mathbf{q}}|^{-2\gamma}+\lambda^2|\hat{\mathbf{q}}|^{2\gamma}-2\delta\big)^{-1}.$$

However, such quantum Hamiltonians are not maximally superintegrable and their spectrum does not convey accidental degeneracy. Therefore these operators define *different* physical systems with respect to the above ones. Nevertheless, maximal superintegrability can be restored in the LB quantization by considering the so-called conformal LB operator Δ_{CLB},

$$\Delta_{\mathrm{CLB}}=\Delta_{\mathrm{LB}}-\frac{(N-2)}{4(N-1)}\,R(|\mathbf{q}|),$$

where R is the scalar curvature (1.2). Then it can be shown that [8]

$$\hat{\mathcal{H}}_{\mathrm{CLB,I}}=-\tfrac{1}{2}\hbar^2\Delta_{\mathrm{CLB}}+A\Big(|\hat{\mathbf{q}}|^{-\beta}-\kappa|\hat{\mathbf{q}}|^{\beta}\Big),$$
$$\hat{\mathcal{H}}_{\mathrm{CLB,II}}=-\tfrac{1}{2}\hbar^2\Delta_{\mathrm{CLB}}+B\big(|\hat{\mathbf{q}}|^{-2\gamma}+\lambda^2|\hat{\mathbf{q}}|^{2\gamma}-2\delta\big)^{-1},$$

are superintegrable Hamiltonians and present accidental degeneracy.

To end with, we stress that the Schrödinger and the conformal LB quantizations are related by means of a gauge transformation. Thus they have the same spectrum and differ in the explicit expressions of the eigenfunctions. In particular, if we consider the Schrödinger equations $\hat{\mathcal{H}}\Phi = E\Phi$ and $\hat{\mathcal{H}}_{\rm CLB}\Phi_{\rm CLB} = E\Phi_{\rm CLB}$ for types I and II, then [8]

$$\hat{\mathcal{H}}_{\rm CLB} = f(|\hat{\mathbf{q}}|)^{(2-N)/2}\hat{\mathcal{H}} f(|\hat{\mathbf{q}}|)^{(N-2)/2}, \quad \Phi_{\rm CLB} = f(|\hat{\mathbf{q}}|)^{(2-N)/2}\Phi,$$

where we have used the conformal factor of the metrics (2.6) and (2.7), namely,

$$f_{\rm I}(|\hat{\mathbf{q}}|)^2 = \frac{1}{\hat{\mathbf{q}}^2\left(|\hat{\mathbf{q}}|^{-\beta} + \kappa|\hat{\mathbf{q}}|^{\beta}\right)^2}, \quad f_{\rm II}(|\hat{\mathbf{q}}|)^2 = \frac{\left(|\hat{\mathbf{q}}|^{-2\gamma} + \lambda^2|\hat{\mathbf{q}}|^{2\gamma} - 2\delta\right)}{\hat{\mathbf{q}}^2\left(|\hat{\mathbf{q}}|^{-2\gamma} - \lambda^2|\hat{\mathbf{q}}|^{2\gamma}\right)^2}.$$

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